On Principal and Non-principal Solutions

Let $p(x), q(x)$ be continuous functions defined on an interval $I = [a, b)$, $b \leq \infty$, $p(x) > 0$, and consider the equation

$$(pu')' + qu = 0 \,. \tag{1}$$

Assume (1) is non-oscillatory. A solution u will be called *principal* if

$$\int^b \frac{dx}{pu^2} = \infty \,, \tag{2}$$

and *non-principal* otherwise. It is understood that the integral in (2) is to be taken over a subinterval $[c, b)$ not containing any zeros of u. There are two main properties about such solutions we would like to derive:

(i) a principal solution u_0 always exists and is unique up to constant multiples;
(ii) if u_1 is non-principal then there are cosntants α, β with $\alpha \neq 0$ such that

$$\frac{u_1}{u_0}(x) = \alpha \int^x \frac{dy}{pu_0^2} + \beta \,. \tag{3}$$

In particular, $|\,(u_1/u_0)\,(x)| \to \infty$ as $x \to b$.

The (invertible) change of variable

$$t = \int_a^x \frac{dy}{p(y)} \tag{4}$$

transforms (1) into

$$v'' + hv = 0 \,, \tag{5}$$

where $h(t) = p(x(t))q(x(t))$. The new interval is $[0, c)$, with $c = \int_a^b (1/p)\, dx$. Solutions of (5) are of the form $v(t) = u(x(t))$ for some solution of (1), and therefore (5) is also non-oscillatory.

We will establish properties $(i), (ii)$ for equation (5), and will show that principal and non-principal solutions of equations (1) and (5) correspond to each other.

It is known that if v_1, v_2 are linearly independent solutions of (5), then

$$F = \frac{v_2}{v_1}$$

is an injective mapping of the interval $[0, c)$ into the extended real line with Schwarzian derivative $SF = 2h$. The group of Möbius transformations T of the real line acts on F by $T(F) = (AF + B)/(CF + D),\ AD - BC \neq 0$, leaving the Schwarzian invariant and changing v_1, v_2 to another set of independent solutions $Av_2 + Bv_1,\ Cv_2 + Dv_1$. Any two mappings with the same Schwarzian will differ by a Möbius trasnformation.

On the other hand, variation of parameters provides a linearly independent solution from a given one v_1 via the fromula

$$v_2 = v_1 \int^t \frac{ds}{v_1^2}\,.$$

Consequently, the function

$$F(t) = \int^t \frac{ds}{v_1^2}$$

has Schwarzian $SF = 2h$. Conversely, any function F with $SF = 2h$ yields the solution $v = (F')^{-1/2}$ of (5).

Let v_1 be any non-principal solution of (5). Since then $F(c) < \infty$, the Möbius change

$$G(t) = \frac{1}{F(c) - F(t)} \tag{6}$$

will now have $G(c) = \infty$. Therefore $v_0 = (G')^{-1/2}$ is a principal solution. This establishes (i) for equation (5).

Let v_1 be any solution of (5), and let

$$w(\tau) = \frac{v_1}{v_0}(H(\tau))\,,$$

where $t = H(\tau) = G^{-1}(\tau)$. Note that $dt/d\tau = v_0^2$. Since $G(c) = \infty$, the function w is defined on some interval $[\tau_0, \infty)$. Direct differentiation shows that $w'' = 0$, hence

$$w(\tau) = \alpha\tau + \beta\,. \tag{7}$$

If v_1 is linearly independent from v_0, then $\alpha \neq 0$, hence

$$\int^c \frac{dt}{v_1^2} = \int^\infty \left(\frac{v_0}{v_1}\right)^2 d\tau = \int^\infty \frac{d\tau}{(\alpha\tau + \beta)^2} < \infty\,.$$

Equation (7) also shows that for a non-principal solution v_1, the quotient $v_1/v_0 \to \infty$ as $t \to c$, growing at a linear rate with respect to the variable $\tau = G(t)$. This proves (ii) for equation (5).

The uniqueness part in (i) as well as equation (3) can be also derived from properties of Möbius transformation. For the uniqueness, if $(G_0')^{-1/2}, (G_1')^{-1/2}$ are principal solutions, then $G_1 = T(G_0)$ for some Möbius transformation that fixes the point at infinity. Hence T is linear, showing that the principal solutions are a constant multiple of each other. For (3), if $(F')^{-1/2}$ is linearly independent from a principal solution $(G')^{-1/2}$, then $F = (AG+B)/(CG+D)$ for some $C \neq 0$. Hence

$$(F')^{-1/2} = (AD-BC)^{-1/2}(CG+D)(G')^{-1/2}.$$

We return to equation (1). Let $u(x) = v(x(t))$ under the change of variables (4). Then

$$\int^b \frac{dx}{pu^2} = \int^c \frac{dt}{v^2},$$

which shows that principal and non-principal solutions of (1) and (5) stand in correspondence. Also, if u_0 is a principal and u_1 an arbitrary solution of (1) then

$$\frac{u_1}{u_0}(x) = \frac{v_1}{v_0}(t(x)) = \alpha G(t(x)) + \beta = \alpha \int^{t(x)} \frac{ds}{v_0^2} + \beta = \alpha \int^x \frac{dy}{pu_0^2} + \beta.$$

We finish with two observations. First, as we showed, when v_1 is a non-principal solution for (5) then

$$v_0 = (G')^{-1/2} = (F')^{-1/2}\,(F(c) - F(t)) = v_1 \int_t^c \frac{ds}{v_1^2}$$

is principal. In other words, if u_1 is non-principal for (1) then

$$u_1 \int_x^b \frac{dy}{pu_1^2}$$

will be principal. On the other hand, let v be any solution of (5) that does not vanish on $[t_0, c)$, and let

$$F(t) = \int_{t_0}^t \frac{ds}{v^2}.$$

Then

$$H = -\frac{1}{F}$$

is increasing and has $H(c) < \infty$. It follows that

$$v_1 = (H')^{-1/2} = (F')^{-1/2} F = v \int_{t_0}^{t} \frac{ds}{v^2}$$

is a non-principal solution of (5). Rephrasing, if u is any solution of (1) which is non-vanishing on $[x_0, b)$ then

$$u \int_{x_0}^{x} \frac{dy}{pu^2}$$

will be non-principal.